Infinite dimension reflection matrices in the sine-Gordon model with a boundary

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ABSTRACT

Using the sine-Gordon model as the prime example an alternative approach to integrable boundary conditions for a theory restricted to a half-line is proposed. The main idea is to explore the consequences of taking into account the topological charge residing on the boundary and the fact it changes as solitons in the bulk reflect from the boundary. In this context, reflection matrices are intrinsically infinite dimensional, more general than the two-parameter Ghoshal-Zamolodchikov reflection matrix, and related in an intimate manner with defects.

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1 Introduction

The investigation of two-dimensional classical and quantum integrable field theories with one or two boundaries is now several decades old (see for example [1]-[9]), yet it is incomplete with many interesting outstanding questions remaining to be answered, even in the context of the much-studied affine Toda models. More recently, there has been some interest in the definition and properties of integrable defects (see, for example [10]-[25]), which is really the study of allowable discontinuities, in the sense of permitting field discontinuities that do not upset integrability, and already there have been suggestions [15, 18, 20] concerning possible relationships between the two ideas. The purpose of this article is to add to these suggestions having in mind models that allow solitons and using the sine-Gordon model as the prime example.

Basically, a defect is a bulk phenomenon but placing a defect close to, or even at, a boundary will effectively modify the reflection matrix associated with the boundary in several ways. First, the defect is able to introduce additional parameters, and second, even if in the absence of the defect a reflection matrix contained no explicit dependence on the topological charge residing at the boundary - since boundary conditions only exceptionally preserve topological charges carried by solitons - the defect will (almost) inevitably introduce dependence on topological charge in the modified reflection matrix. So, it would seem sensible to reconsider the boundary Yang-Baxter equation, or reflection equation, allowing the boundary to carry topological charge, and the reflection matrix to be infinite-dimensional, then attempt to classify its solutions.

If the sine-Gordon model is considered as the primary example, on the grounds that it is the simplest of the affine Toda field theories, one might expect the solutions to the reflection equation to be one of three types: (1) the coefficients of the reflection matrix could be independent of topological charge (basically, this is the Ghoshal-Zamolodchikov solution [4], though reformulated slightly); (2) the reflection matrix is related to a Ghoshal-Zamolodchikov solution modified by a defect; or (3) neither of these - in which case the reflection matrix may describe a new type of boundary condition, which might turn out to be a defect 'fused' with a boundary. The latter idea might appear strange at first sight yet it is worth recalling that defects themselves may be fused together and the essential difference between an unfused and a fused pair is the evident fact that a soliton can propagate in the gap between the unfused pair and there is no gap between a fused pair [22]. One need look no further than a Neumann boundary condition (placing the boundary at $x = 0$), $u_x(0,t) = 0$, to realise that a boundary can store an unlimited quantity of topological charge since N solitons approaching the boundary eventually reflect as N anti-solitons, or vice versa, with the boundary picking up $2N$ units of charge. In effect, the boundary value of u changes by $4\pi N$ during the process. On the other hand, a Dirichlet boundary condition, for example $u(0,t) = u_0$, ensures solitons reflect as solitons and the boundary charge does not change during the process.

Under reasonably general assumptions, effectively meaning the absence of any dynamical variables confined to the boundary, classically integrable boundary conditions have been determined [5] for any of the (real) affine Toda field theories in the following form

$$u_x(0,t) = -\frac{\partial \mathcal{B}}{\partial u}, \quad \mathcal{B} = \sum_{r=0}^{n} b_r\, e^{\alpha_r \cdot u(0,t)/2}, \tag{1.1}$$

where u is a multicomponent field, the vectors α_r are the Euclidean parts of the affine roots

described by any of the Dynkin-Kac diagrams [26], the coefficients b_r are constants. These boundary conditions possess curious properties [6]. With a single exception, the most unexpected concerns the affine Toda field theories based on $a_n^{(1)}$, $d_n^{(1)}$, $e_n^{(1)}$ root data. Apart from the sine-Gordon model (the exception, based on $a_1^{(1)}$ data), which has two free parameters b_1, b_2 associated with a general integrable boundary condition, these models do not appear to allow any free parameters associated with a boundary. Instead, there is a discrete set of non-zero values for the coefficients b_r amounting to a set of sign choices (Neumann is always a possibility with $b_r = 0$ for all r). The other models (based on non-simply-laced data) fare a little better, in the sense that there are usually one or two free parameters but never as many as the rank of the associated affine algebra. In the complex affine Toda models (where there are complex solitons), there are other boundary conditions [7] of the form

$$\mathrm{Im}\, u(0,t) = 2\pi\lambda, \quad \mathrm{Re}\, u_x(0,t) = 0, \tag{1.2}$$

where λ is a weight. Clearly these are a mixture of Dirichlet and Neumann conditions, and they are soliton-preserving. In the complex models the conditions (1.1) are not soliton-preserving and will inevitably change the topological charge on the boundary. It is possible that the assumptions made to reach these results are too stringent but it has not yet been found how to relax them. It is also possible that dressing a boundary with one or more defects may add additional parameters and provide a clue towards the goal of generalising the boundary conditions. On the other hand, this is easier said than done because, so far, defects are only established for the affine Toda models based on $a_n^{(1)}$, $n \geq 1$ and $a_2^{(2)}$ [21, 23].

2 Generalised reflection matrices

Consider the sine-Gordon model, a typical reflection matrix is

$$R_{a\,\alpha}^{b\,\beta}(\theta), \tag{2.1}$$

where the labels $a, b = \pm 1$ (or simply $\pm$) represent the incoming and outgoing soliton, the labels α, β represent the initial and final topological charge carried by the boundary, and θ is the rapidity of the incoming soliton. In much of the earlier literature the boundary topological charge labels are simply suppressed but here it will be useful to keep them explicit. Clearly, topological charge conservation requires

$$a + \alpha = b + \beta, \tag{2.2}$$

and the boundary labels must be either even or odd since they may only change by ± 2. The reflection matrix will also depend upon the bulk coupling and boundary parameters. In the more general affine Toda models, both sets of labels should be replaced by suitable sets of weights.

The reflection equation, or boundary Yang-Baxter equation [1], is a condition that ensures compatibility with the bulk soliton S-matrix. Thus

$$R_{a\,\alpha}^{q\,\beta}(\theta_a)\, S_{b\,q}^{p\,s}(\theta_b+\theta_a) R_{p\,\beta}^{r\,\gamma}(\theta_b)\, S_{s\,r}^{d\,c}(\theta_b-\theta_a) = S_{b\,a}^{p\,q}(\theta_b-\theta_a) R_{p,\alpha}^{r\,\beta}(\theta_b) S_{q\,r}^{s\,c}(\theta_a+\theta_b) R_{s\,\beta}^{d\,\gamma}(\theta_a), \tag{2.3}$$

where repeated indices are summed. In the sine-Gordon model, the bulk S-matrix [27] is given by a 4×4 matrix depending on the parameter $\Theta \equiv \Theta_{12} = (\theta_1 - \theta_2)$ whose elements different from zero are:

$$\begin{aligned}
S_{++}^{++}(\Theta) &= S_{--}^{--}(\Theta) = \left(\frac{qx_1}{x_2} - \frac{x_2}{qx_1}\right)\rho_s(\Theta) \equiv a(\Theta)\,\rho_s(\Theta),\\
S_{+-}^{-+}(\Theta) &= S_{-+}^{+-}(\Theta) = \left(\frac{x_1}{x_2} - \frac{x_2}{x_1}\right)\rho_s(\Theta) \equiv b(\Theta)\,\rho_s(\Theta),\\
S_{+-}^{+-}(\Theta) &= S_{-+}^{-+}(\Theta) = \left(q - \frac{1}{q}\right)\rho_s(\Theta) \equiv c\,\rho_s(\Theta),
\end{aligned} \tag{2.4}$$

with

$$x_p = e^{\gamma\theta_p}, \quad q = -e^{-i\pi\gamma} = e^{-4i\pi^2/\beta^2}, \quad \gamma = \frac{4\pi}{\beta^2} - 1.$$

The multiplicative factor $\rho_s(\Theta)$ is given by:

$$\rho_S(\Theta) = \frac{\Gamma(1-z-\gamma)\Gamma(1+z)}{2\pi i}\prod_{k=1}^{\infty} R_k(\Theta)R_k(i\pi - \Theta), \quad z = \frac{i\gamma\Theta}{\pi}, \tag{2.5}$$

with

$$R_k(\Theta) = \frac{\Gamma(z+2k\gamma)\Gamma(1+z+2k\gamma)}{\Gamma(z+(2k+1)\gamma)\Gamma(1+z+(2k-1)\gamma)}.$$

There are many solutions to (2.3) that may be written

$$R_{a\,\alpha}^{b\,\beta}(\theta) = \begin{pmatrix} r_+(\alpha,x)\,\delta_\alpha^\beta & s_+(\alpha,x)\,\delta_\alpha^{\beta-2} \\ s_-(\alpha,x)\,\delta_\alpha^{\beta+2} & r_-(\alpha,x)\,\delta_\alpha^\beta \end{pmatrix}, \tag{2.6}$$

where the reflection factor is written in block form, and it is understood the blocks are labeled by the solitons. Typically, the coefficients appearing in (2.6) will be functions of α, the initial charge on the boundary, besides the rapidity, bulk coupling and boundary parameters. However, there is a particular two-parameter class of solutions, referred to as the Ghoshal-Zamolodchikov solution, where there is no dependence on α other than via the Kronecker deltas. In this case the latter are there merely to keep track of the topological charge. This solution has the following form,

$$R_{a\,\alpha}^{b\,\beta}(\theta) = \sigma(\theta)\begin{pmatrix} (r_1x + r_2/x)\,\delta_\alpha^\beta & k_0\,(x^2 - 1/x^2)\,\delta_\alpha^{\beta-2} \\ l_0\,(x^2-1/x^2)\,\delta_\alpha^{\beta+2} & (r_2x + r_1/x)\,\delta_\alpha^\beta \end{pmatrix}, \tag{2.7}$$

where r_1, r_2, k_0, l_0 are constants and $x = e^{\gamma\theta}$. Without losing generality one may impose $r_1r_2 = 1$, $k_0 = l_0$, by removing an overall constant factor and using a similarity transformation. The requirements of crossing and unitarity restrict the overall scalar factor $\sigma(\theta)$ [4].

A more general solution to (2.3) can be obtained quite easily by assuming the elements $s_\pm(\alpha,x)$ remain proportional to $x^2 - 1/x^2$ and the diagonal elements are proportional to a cubic in x and $1/x$. Explicitly, apart from an overall function of rapidity, the solution is:

$$\begin{gathered}
r_+(\alpha,x) = \left(x^2 - 1/x^2\right)\left(r_3q^{\alpha+1}x - r_4q^{-\alpha-1}/x\right) + r_1x + r_2/x,\\
r_-(\alpha,x) = \left(x^2 - 1/x^2\right)\left(r_4q^{-\alpha+1}x - r_3q^{\alpha-1}/x\right) + r_2x + r_1/x,\\
s_+(\alpha,x) = \left(x^2 - 1/x^2\right)(k_0 + k_1q^\alpha + k_2q^{-\alpha}), \quad s_-(\alpha,x) = \left(x^2 - 1/x^2\right)(l_0 + l_1q^\alpha + l_2q^{-\alpha})\\
k_1l_1 = -r_3^2, \quad k_2l_2 = -r_4^2, \quad k_1l_0 + q^2k_0l_1 = qr_2r_3, \quad k_0l_2 + q^2k_2l_0 = qr_1r_4.
\end{gathered} \tag{2.8}$$

Clearly, the expression (2.7) lies within this class of solution and is obtained by setting $r_3 = r_4 = 0 = k_1 = k_2 = l_1 = l_2$. Another interesting special case concerns the choice $r_1 = r_2 = 0$ for which the quantity $(x^2 - 1/x^2)$ becomes an overall factor. Actually, assuming all terms in (2.6) are polynomial in x, it is not difficult to show that the polynomials must have common zeros, except for the roots of $x^4 = 1$, which may be present in the off-diagonal terms without also being roots of the diagonal terms. For this reason, the solution given above is general. More detail concerning these solutions is supplied in appendix A.

If a defect is placed before the boundary this will modify a reflection factor though not merely by altering its dependence on rapidity. Since a defect also carries topological charge a new set of labels will be added to the modified reflection matrix [2, 15]. Thus, in detail,

$$R^{b\,\beta\,\tilde{\beta}}_{a\,\alpha\,\tilde{\alpha}}(\theta) = T^{c\,\tilde{\gamma}}_{a\,\tilde{\alpha}}(\theta) R^{d\,\beta}_{c\,\alpha}(\theta) \hat{T}^{b\,\tilde{\beta}}_{d\,\tilde{\gamma}}(\theta), \tag{2.9}$$

where $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ are topological charges associated with the defect, T represents the transmission matrix for the soliton approaching the boundary, while $\hat{T}$ represents the transmission matrix for the soliton after reflection from the boundary. In fact $\hat{T}(\theta)$ is the inverse of $T(-\theta)$. The combination on the right hand side of (2.9) automatically satisfies a suitable generalisation of the compatibility relations (2.3), provided the transmission matrix is also compatible with the bulk soliton S-matrix. The latter requires,

$$S^{mn}_{ab}(\theta_a - \theta_b)\, T^{t\beta}_{n\alpha}(\theta_a)\, T^{s\gamma}_{m\beta}(\theta_b) = T^{n\beta}_{b\,\alpha}(\theta_b)\, T^{m\gamma}_{a\,\beta}(\theta_a)\, S^{st}_{mn}(\theta_a - \theta_b). \tag{2.10}$$

If the reflection matrix on the right hand side of (2.9) is proportional to δ^{β}_{α} then the new reflection matrix on the left hand side is also proportional to δ^{β}_{α} and satisfies precisely (2.3).

In the sine-Gordon model solutions to (2.10) are known and may therefore be used to construct solutions to (2.3). Owing to topological charge conservation, they all possess the general shape

$$T^{b\,\beta}_{a\,\alpha}(\theta) = \rho(\theta) \begin{pmatrix} a(\alpha, x)\, \delta^{\beta}_{\alpha} & b(\alpha, x)\, \delta^{\beta-2}_{\alpha} \\ c(\alpha, x)\, \delta^{\beta+2}_{\alpha} & d(\alpha, x)\, \delta^{\beta}_{\alpha} \end{pmatrix}. \tag{2.11}$$

In [22], the Konik-LeClair solution [11] to (2.10) was generalised to

$$T^{b\,\beta}_{a\,\alpha}(\theta) = \rho(\theta) \begin{pmatrix} (a_+ q^{-\alpha/2} x^{-1} + a_- q^{\alpha/2} x)\, \delta^{\beta}_{\alpha} & \mu(\alpha)\, \delta^{\beta-2}_{\alpha} \\ \lambda(\alpha)\, \delta^{\beta+2}_{\alpha} & (d_+ q^{-\alpha/2} x + d_- q^{\alpha/2} x^{-1})\, \delta^{\beta}_{\alpha} \end{pmatrix}, \tag{2.12}$$

with

$$\mu(\alpha)\, \lambda(\alpha+2) - \mu(\alpha-2)\, \lambda(\alpha) = (q - q^{-1})\, (a_- d_- q^{\alpha} - a_+ d_+ q^{-\alpha}),$$

which implies

$$\mu(\alpha-2)\, \lambda(\alpha) = a_- d_- q^{\alpha-1} + a_+ d_+ q^{-\alpha+1} + \gamma. \tag{2.13}$$

Actually, it has not proved possible to generalise (2.12) further and it is most probably already the general solution (see also [28]). The constraint (2.13) is satisfied by choosing, for instance,

$$\mu(\alpha) = b_+ q^{-\alpha/2} + b_- q^{\alpha/2}, \quad \lambda(\alpha) = c_+ q^{-\alpha/2} + c_- q^{\alpha/2}, \quad a_\pm\, d_\pm - b_\pm\, c_\pm = 0,$$

where $a_\pm$, $b_\pm$, $c_\pm$, $d_\pm$ are otherwise free (complex) constants. Then, setting $Q = q^{-\alpha/2}$, the expression (2.12) becomes

$$T^{b\,\beta}_{a\,\alpha}(\theta) = \rho(\theta) \begin{pmatrix} (a_+ Q^{\alpha} x^{-1} + a_- Q^{-\alpha} x)\, \delta^{\beta}_{\alpha} & (b_+ Q^{\alpha} + b_- Q^{-\alpha})\, \delta^{\beta-2}_{\alpha} \\ (c_+ Q^{\alpha} + c_- Q^{-\alpha})\, \delta^{\beta+2}_{\alpha} & (d_+ Q^{\alpha} x + d_- Q^{-\alpha} x^{-1})\, \delta^{\beta}_{\alpha} \end{pmatrix}. \tag{2.14}$$

Its inverse is given by:

$$\left(T^{-1}\right)^{b\,\beta}_{a\,\alpha} = \frac{1}{\Delta\rho}\left(\begin{array}{cc} (d_+Q^{-\alpha-2}\,x + d_-Q^{\alpha+2}x^{-1})\,\delta^{\beta}_{\alpha} & -(b_+Q^{-\alpha} + b_-Q^{\alpha})\,\delta^{\beta-2}_{\alpha} \\ -(c_+Q^{-\alpha} + c_-Q^{\alpha})\,\delta^{\beta+2}_{\alpha} & (a_+Q^{-\alpha+2}\,x^{-1} + a_-Q^{\alpha-2}\,x)\,\delta^{\beta}_{\alpha} \end{array}\right), \qquad (2.15)$$

where

$$\Delta = a_-d_+qx^2 + a_+d_-q^{-1}x^{-2} - b_+c_-q^{-1} - b_-c_+q.$$

Take the reflection matrix appearing on the right hand side of (2.9) to be diagonal, for example a Ghoshal-Zamolodchikov solution (2.7) with $r_1 = 1/r_2 \equiv r$, $k_0 = l_0 = 0$, corresponding to a Dirichlet boundary condition,

$$R^{d\,\beta}_{c\,\alpha}(\theta) = \sigma(\theta)\left(\begin{array}{cc} (rx + x^{-1}r^{-1})\,\delta^{\beta}_{\alpha} & 0 \\ 0 & (rx^{-1} + xr^{-1})\,\delta^{\beta}_{\alpha} \end{array}\right). \qquad (2.16)$$

Then, the result of the calculation (2.9) has the general shape (2.6) with coefficients of the following form (up to an overall factor)

$$\begin{aligned} r_+(\alpha,x) &= (x^2 - x^{-2})\,(a_-d_-q^{\alpha+1}rx - a_+d_+q^{-\alpha-1}r^{-1}x^{-1}) + x\,(gr - hr^{-1}) + x^{-1}\,(gr^{-1} - hr)\,, \\ r_-(\alpha,x) &= (x^2 - x^{-2})\,(a_+d_+q^{-\alpha+1}r^{-1}x - a_-d_-q^{\alpha-1}rx^{-1}) + x\,(gr^{-1} - hr) + x^{-1}\,(gr - hr^{-1})\,, \\ s_+(\alpha,x) &= (x^2 - x^{-2})\,(a_+b_-r^{-1} - a_-b_+r - a_-b_-q^{\alpha}r + a_+b_+q^{-\alpha}r^{-1})\,, \\ s_-(\alpha,x) &= (x^2 - x^{-2})\,(d_-c_+r - d_+c_-r^{-1} + d_-c_-q^{\alpha}r - d_+c_+q^{-\alpha}r^{-1})\,, \end{aligned} \qquad (2.17)$$

with

$$g = a_-d_+q^{-1} + a_+d_-q, \qquad h = b_-c_+q^{-1} + b_+\,c_-q.$$

The Konik-LeClair [11] (type I) solutions are a particular case of (2.14). Set, for example, $a_- = d_+ = c_- = b_+ = 0$, multiply (2.14) by x and collect an overall factor. Then, (2.14) becomes

$$T_I\,{}^{b\,\beta}_{a\alpha}(\theta) = \rho_I(\theta)\left(\begin{array}{cc} \nu^{-1/2}\,Q^{\alpha}\,\delta^{\beta}_{\alpha} & x\,\varepsilon\,Q^{-\alpha}\,\delta^{\beta-2}_{\alpha} \\ x\,\varepsilon\,Q^{\alpha}\,\delta^{\beta+2}_{\alpha} & \nu^{1/2}\,Q^{-\alpha}\,\delta^{\beta}_{\alpha} \end{array}\right),$$

with $\nu^{1/2} = (d_-/a_+)^{1/2}$, $b_- = c_+$, $\varepsilon = b_-\,(a_+\,d_-)^{-1/2}$ and ρ_I is an overall factor constrained by the usual requirements of crossing and unitarity. By performing a similarity transformation the α dependence of the off diagonal entries can be eliminated and the type I defect matrix, as written in [22], is recovered. Then, the corresponding coefficients of the R matrix (2.9) are:

$$\begin{aligned} r_+ &= x\,(r\,q - \varepsilon^2r^{-1}q^{-1}) + x^{-1}\,(r^{-1}q - \varepsilon^2rq^{-1})\,, \\ r_- &= x\,(r^{-1}q - \varepsilon^2rq^{-1}) + x^{-1}\,(rq - \varepsilon^2r^{-1}q^{-1})\,, \\ s_+ &= (x^2 - x^{-2})\,\varepsilon r^{-1}\nu^{-1/2}, \\ s_- &= (x^2 - x^{-2})\,\varepsilon r\nu^{1/2}, \end{aligned} \qquad (2.18)$$

which is equivalent to the full Ghoshal-Zamolodchikov solution.

On the other hand, a 'type II' defect is obtained from (2.14) by setting $a_+ = d_- = 1$, $a_- = -\bar{b}_+\,b_-$, $d_+ = -b_+\,\bar{b}_-\,q^2$, $c_+ = -\bar{b}_-\,q^2$, $c_- = -\bar{b}_+$ and multiplying by x. Then the expression in [22] is recovered. That is,

$$T_{II}\,{}^{b\,\beta}_{a\,\alpha}(\theta) = \rho_{II}(\theta)\left(\begin{array}{cc} (Q^{\alpha} - b_-\,\bar{b}_+\,Q^{-\alpha}\,x^2)\,\delta^{\beta}_{\alpha} & x\,(b_+Q^{\alpha} + b_-Q^{-\alpha})\,\delta^{\beta-2}_{\alpha} \\ -x\,(\bar{b}_-\,Q^{\alpha-4} + \bar{b}_+\,Q^{-\alpha})\,\delta^{\beta+2}_{\alpha} & (-b_+\bar{b}_-\,Q^{\alpha-4}\,x^2 + Q^{-\alpha})\,\delta^{\beta}_{\alpha} \end{array}\right),$$

where the form of the scalar function ρ_{II} is also constrained by crossing and unitarity. The corresponding coefficients of the reflection matrix (2.9) are:

$$\begin{aligned} r_+ &= (x^2 - x^{-2})\left(b_+\bar{b}_- Q^{2\alpha-2} r^{-1} x^{-1} - \bar{b}_+ b_- Q^{-2\alpha-2} r x\right) + x\,(gr - hr^{-1}) + x^{-1}\,(gr^{-1} - hr)\,, \\ r_- &= (x^2 - x^{-2})\left(-b_+\bar{b}_- Q^{2\alpha-6} r^{-1} x + \bar{b}_+ b_- Q^{-2\alpha+2} r x^{-1}\right) + x\,(gr^{-1} - hr) + x^{-1}\,(gr - hr^{-1})\,, \\ s_+ &= (x^2 - x^{-2})\left(b_- r^{-1} + |b_+|^2 b_- r + b_+ Q^{2\alpha} r^{-1} + b_-^2 \bar{b}_+ Q^{-2\alpha} r\right), \\ s_- &= (x^2 - x^{-2})\left(\bar{b}_- Q^{-4} r + |b_+|^2 \bar{b}_- Q^{-4} r^{-1} + \bar{b}_-^2 b_+ Q^{2\alpha-8} r^{-1} + \bar{b}_+ Q^{-2\alpha} r\right), \end{aligned} \tag{2.19}$$

with

$$g = \left(|b_+|^2 |b_-|^2 + 1\right) Q^{-2}, \qquad h = -\left(|b_-|^2 Q^{-2} + |b_+|^2\right) Q^{-2}.$$

These entries correspond to a new solution of the reflection equation (2.3), in which the explicit dependence on the label α cannot be removed by a similarity transformation.

It is tempting to suggest, given the Lagrangian description of a type II defect [21, 22], that the system associated with this reflection matrix is described classically (dropping explicit reference to mass scale and the bulk coupling constant), by the following Lagrangian density

$$\mathcal{L}_B(u,\lambda) = \theta(-x)\,\mathcal{L}_u + \delta(x)(u\lambda_t - B(u,\lambda)), \qquad \mathcal{L}_u = u_t^2/2 - u_x^2/2 - \left(e^u + e^{-u}\right), \tag{2.20}$$

where u is the sine/sinh-Gordon field (depending on whether u is imaginary or real), λ is a time-dependent field defined at the boundary and $B(u,\lambda)$ is a functional to be determined. Forcing the energy-like spin three charge to be conserved (using the argument appearing first in [4]), the form of the functional B is constrained to be:

$$B(u,\lambda) = e^{\lambda/2} f(u) + e^{-\lambda/2} g(u), \tag{2.21}$$

with

$$f(u)g(u) = h_+ e^{u/2} + h_- e^{-u/2} + 2\,(e^u + u^{-u}) + h_0, \tag{2.22}$$

where $h_\pm$, h_0 are free constant parameters. Since redefining $\lambda \to \lambda + \Lambda(u)$ changes the Lagrangian by a total derivative, it follows that the boundary part of the Lagrangian density is essentially a three parameter expression, which can be represented conveniently by (2.20) and (2.21) on choosing

$$f(u) = f_0 + \sqrt{2}(b e^{u/2} + b^{-1} e^{u/2}), \qquad g(u) = g_0 + \sqrt{2}(b^{-1} e^{u/2} + b e^{u/2}). \tag{2.23}$$

Notice that the most general solution of (2.22) has an additional parameter. However, when the expressions of the functionals f and g are used in (2.21) one parameter can be eliminated by rescaling the field λ. It can be verified that this expression for the boundary potential corresponds to the expression obtained by adding together a type II defect potential with the Ghoshal-Zamolodchikov Dirichlet boundary potential (see [18], [22] for similar arguments).

3 Discussion

Some comments are in order at this point. A defect can be used to create solutions to the reflection Yang-Baxter equation (2.3) via (2.9). Using the reflection matrix (2.16) representing

any Dirichlet boundary condition and the transmission matrix of a type I defect the Ghoshal-Zamolodchikov solution (2.7) is recovered in (2.18). In this case the reflection matrix has no dependence on topological charge other than via the Kronecker deltas. Note, the simplest Dirichlet boundary condition imposes $u(0,t)=0$, and its reflection factor has the form (2.16) with $r=1$; it is proportional to the identity. This particular boundary condition preserves the bulk symmetry $u\to -u$ and, using a type I defect, the solution of (2.3) is (2.18) with $r=1$. In this case, the ensuing nonlinear boundary condition also maintains the bulk symmetry $u\to -u$.

Also note, the results are consistent with an interesting observation by Habibullin [8] who pointed out how in the classical sine-Gordon theory a general boundary condition could be obtained from a general Dirichlet condition (that is, one with $u_0\neq 0$), using a combination of two carefully designed Bäcklund transformations. Explicitly, the combined effect of the two Bäcklund transformations is effectively identical to the Ghoshal-Zamolodchikov general boundary condition, namely,

$$u_x(0,t)=\kappa\sin\frac{(u+u_0)}{2}+\kappa^{-1}\sin\frac{(u-u_0)}{2},$$

where κ is the parameter introduced by the Bäcklund transformations. It was pointed out also some time ago [13] that a type I defect is, from a classical point of view, a Bäcklund transformation 'frozen' at the location of the defect. Therefore, the action of the defect in modifying a boundary condition is essentially the Habibullin construction and (2.9) is the quantum field theoretic version of this. Bajnok and Simon [18] noted the same phenomenon, albeit stated differently, by adding a type I defect to a Dirichlet condition in the sinh-Gordon model.

Using a type II defect to generate solutions of (2.3) leads inevitably to solutions that are outside the Ghoshal-Zamolodchikov class since the explicit dependence of their coefficients on topological charge is unavoidable. One may wonder what boundary conditions might be responsible for these and one suggestion has been proposed with the boundary Lagrangian density (2.20). The most general solution (2.8) to the transmission Yang-Baxter equation in the form (2.3) has been derived explicitly and a comprehensive explanation concerning the manner in which it is obtained using defects has been provided. One could imagine that repeatedly placing type I or type II defects near a Dirichlet boundary, or near a general Ghoshal-Zamolodchikov boundary, should generate all possible solutions to a (suitably generalised) version of the compatibility relations (2.3) containing additional pairs of topological charge labels (for example, see (2.9)). However, the types of classical boundary condition to which they might effectively correspond (if any) is unclear. One should bear in mind that any process of reduction (or 'fusing'), in which the additional sets of labels collapse to a single set, must yield the already known general solutions provided above, or a combination of them. On the other hand, the most general solution to (2.10) (with a single pair of topological charge labels) is provided by (2.12) (see also [28] for an alternative algebraic argument). Hence there is no other solution to (2.10) that could be used in combination with a Dirichlet boundary condition to provide a more general solution to (2.3). It is worth recalling that boundaries with additional degrees of freedom have been considered previously for the sine-Gordon model by Baseilhac and Delius [29] and the corresponding reflection matrix found in [30]. However,the latter represents a solution of a boundary Yang-Baxter equation that does not preserve the topological charge of the system. Hence it is not of the form (2.3).

Finally, generalising to the complex affine Toda models, the analogue of the Dirichlet condition

is given in (1.2) and it would be interesting to discover how the reflection matrices corresponding to these are related (if indeed they are) to the reflection matrices associated with the boundary conditions (1.1). The purpose of this note was to emphasise the role defects might play and these specific questions will be addressed in the future.

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A Infinite dimensional solution to the reflection Yang-Baxter equation

Given the expression (2.6) for the reflection matrix the reflection Yang-Baxter equation becomes a collection of nonlinear recurrence relations for the coefficients. These are of four basic types, 2-term, 4-term, 5-term and 6-term relations. It is also useful to define $a^{\pm}(x_1,x_2)=a(\theta_2\pm\theta_1)$, $b^{\pm}(x_1,x_2)=b(\theta_2\pm\theta_1)$.

The 2-term relations involve just the off-diagonal elements and are:

$$\begin{aligned}s_{\pm}(\alpha,x_1)s_{\pm}(\alpha+2,x_2)&=s_{\pm}(\alpha,x_2)s_{\pm}(\alpha+2,x_1),\\ s_{+}(\alpha,x_1)s_{-}(\alpha+2,x_2)&=s_{+}(\alpha,x_2)s_{-}(\alpha+2,x_1),\end{aligned}\tag{A.1}$$

implying $s_{\pm}(\alpha,x)=f(x)s_{\pm}(\alpha)$, where $f(x)$ depends on rapidity but not topological charge and the coefficients $s_{\pm}(\alpha)$ do not depend on rapidity.

There are a number of 4-term relations all of the following type,

$$\begin{aligned}&s_{+}(\alpha,x_2)\left(a^{-}b^{+}\ r_{+}(\alpha+2,x_1)-a^{+}b^{-}\ r_{+}(\alpha,x_1)\right)\\ &\qquad=cs_{+}(\alpha,x_1)\left(b^{+}\ r_{+}(\alpha+2,x_2)-b^{-}\ r_{-}(\alpha+2,x_2)\right),\end{aligned}\tag{A.2}$$

which simplify to

$$\begin{aligned}&f(x_2)\left(a^{-}b^{+}\ r_{+}(\alpha+2,x_1)-a^{+}b^{-}\ r_{+}(\alpha,x_1)\right)\\ &\qquad=cf(x_1)\left(b^{+}\ r_{+}(\alpha+2,x_2)-b^{-}\ r_{-}(\alpha+2,x_2)\right),\end{aligned}\tag{A.3}$$

and a number of 5-term relations of the form,

$$\begin{aligned}&s_{+}(\alpha,x_1)\left(a^{+}a^{-}\ r_{+}(\alpha,x_2)-b^{+}b^{-}\ r_{+}(\alpha+2,x_2)-c^{2}r_{-}(\alpha+2,x_2)\right)\\ &\qquad=cs_{+}(\alpha,x_2)\left(a^{+}\ r_{+}(\alpha,x_1)-a^{-}\ r_{-}(\alpha+2,x_1)\right).\end{aligned}\tag{A.4}$$

Again, using (A.1), these simplify to,

$$f(x_1)((a^- a^+ \; r_+(\alpha, x_2) - b^+ b^- \; r_+(\alpha+2, x_2) - c^2 \; r_-(\alpha+2, x_2)) \\ = c f(x_2) \left(a^+ \; r_+(\alpha, x_1) - a^- \; r_-(\alpha+2, x_1)\right). \tag{A.5}$$

Finally, there is a 6-term relation that reads

$$cb^-(r_+(\alpha, x_1) r_+(\alpha, x_2) - r_-(\alpha, x_1) r_-(\alpha, x_2)) \\ + cb^+(r_+(\alpha, x_1) r_-(\alpha, x_2) - r_+(\alpha, x_2) r_-(\alpha, x_1)) \\ + a^+ b^-(s_+(\alpha, x_1) s_-(\alpha+2, x_2) - s_+(\alpha-2, x_1) s_-(\alpha, x_2)) = 0. \tag{A.6}$$

The latter is the only nonlinear relation to supply a real constraint on the off-diagonal functions of α, $s_\pm(\alpha)$. Making the ansatz $f(x) \approx (x^2 - 1/x^2)$ and assuming $r_\pm$ are cubic in x and $1/x$ leads (using Maple) to the solution given in (2.8). In effect equations (A.3) and (A.5) are solved first for $r_\pm$ and the last equation (A.6) strongly constrains $s_\pm(\alpha)$.

More generally, it is useful to consider the zeros of $f(x)$ and compare them with the zeros of $r_\pm(\alpha, x)$. Suppose x_2 is a root of $f(x)$ and x_1 is not a root of $f(x)$. Then, the right hand side of (A.2) becomes

$$b^+ r_+(\alpha, x_2) - b^- r_-(\alpha, x_2) = 0,$$

and therefore, using the explicit forms of $b^\pm$,

$$x_1^2 \left[x_2^2 r_+(\alpha, x_2) + r_-(\alpha, x_2)\right] - \left[x_2^2 r_-(\alpha, x_2) + r_+(\alpha, x_2)\right] = 0.$$

Thus, since this must hold for any choice of x_1,

$$x_2^2 r_+(\alpha, x_2) + r_-(\alpha, x_2) = 0, \quad x_2^2 r_-(\alpha, x_2) + r_+(\alpha, x_2) = 0,$$

which together imply $(1 - x_2^4) r_\pm = 0$. Thus, either x_2 is also a zero of $r_\pm(\alpha, x_2)$, or $x_2^4 = 1$. In other words, the only zeros of $f(x)$ not shared by the diagonal elements are located at the fourth roots of unity.

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